Research Article

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Jensen-type inequalities for *m*-convex functions

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Abstract: Inequalities play an important role in pure and applied mathematics. In particular, Jensen's inequality, one of the most famous inequalities, plays the main role in the study of the existence and uniqueness of initial and boundary value problems for differential equations. In this work, we prove some new Jensen-type inequalities for m-convex functions and apply them to generalized Riemann-Liouville-type integral operators. Furthermore, as a remarkable consequence, some new inequalities for convex functions are obtained.

Keywords: Jensen-type inequalities, convex functions, m-convex functions, fractional derivatives and integrals, fractional integral inequalities

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1 Introduction

As the authors of [1,2,4,6,18,19,29,32] have pointed out, integral inequalities belong to the class of mathematical inequalities playing a significant leading role in different fields of science and technology, such as physical, biological, and engineering sciences, approximation theory and spectral analysis, statistical analysis and theory of distributions, financial economics, and computer science. It is noteworthy that for this last field, and despite the optimization problems in deep learning being generally nonconvex, they often exhibit some properties of convex ones near local minima. This phenomenon allows us to obtain applications of Jensen inequality and some of its generalizations in machine learning (c.f., e.g., [13] and references therein).

In recent years, there has been a growing interest in the study of many classical inequalities applied to integral operators associated with different types of fractional derivatives, since integral inequalities and their applications play a vital role in the theory of differential equations and applied mathematics. Some of the inequalities studied are Gronwall, Chebyshev, Hermite-Hadamard-type, Ostrowski-type, Grüss-type, Hardy-type, Gagliardo-Nirenberg-type, reverse Minkowski, and reverse Hölder inequalities (see, e.g., [3,8,20,21,24–28]).

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We refer the interested reader to the classical books [17,18,23], the monograph [4], and references cited therein for a detailed exposition about the fundamentals of some recent trends of the research in this broad field.

Motivated by some of the aforementioned results and the paper [16], our contributions in the present paper are addressed both to the theory of inequalities and the theory of fractional operators. On one hand, we provide some new Jensen-type inequalities. And, on the other hand, we use such inequalities for obtaining novel results in the current setting of fractional integral inequalities.

2 Preliminaries

One of the first operators that can be called fractional is the Riemann-Liouville fractional derivative of order $\alpha \in \mathbb{C}$, with $\mathrm{Re}(\alpha) > 0$, defined as follows (see [7]).

Definition 1. Let $a < b$ and $f \in L^1((a, b); \mathbb{R})$. The *right and left side Riemann-Liouville fractional integrals of order* α, with $\mathrm{Re}(\alpha) > 0$, are defined, respectively, by

$$^{RL}J^{\alpha}_{a^+}f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}f(s)\mathrm{d}s \tag{1}$$

and

$$^{RL}J^{\alpha}_{b^-}f(t) = \frac{1}{\Gamma(\alpha)}\int_t^b (s-t)^{\alpha-1}f(s)\mathrm{d}s, \tag{2}$$

with $t \in (a, b)$.

When $\alpha \in (0, 1)$, their corresponding *Riemann-Liouville fractional derivatives* are given by

$$({}^{RL}D^{\alpha}_{a^+}f)(t) = \frac{\mathrm{d}}{\mathrm{d}t}({}^{RL}J^{1-\alpha}_{a^+}f(t)) = \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^t \frac{f(s)}{(t-s)^{\alpha}}\mathrm{d}s$$

$$({}^{RL}D^{\alpha}_{b^-}f)(t) = -\frac{\mathrm{d}}{\mathrm{d}t}({}^{RL}J^{1-\alpha}_{b^-}f(t)) = -\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_t^b \frac{f(s)}{(s-t)^{\alpha}}\mathrm{d}s.$$

Other definitions of fractional operators are the following ones.

Definition 2. Let $a < b$ and $f \in L^1((a, b); \mathbb{R})$. The *right and left side Hadamard fractional integrals of order* α, with $\mathrm{Re}(\alpha) > 0$, are defined, respectively, by

$$H^{\alpha}_{a^+}f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{f(s)}{s}\mathrm{d}s \tag{3}$$

and

$$H^{\alpha}_{b^-}f(t) = \frac{1}{\Gamma(\alpha)}\int_t^b \left(\log\frac{s}{t}\right)^{\alpha-1}\frac{f(s)}{s}\mathrm{d}s, \tag{4}$$

with $t \in (a, b)$.

When $\alpha \in (0,1)$, *Hadamard fractional derivatives* are given by the following expressions:

$$({}^H D^{\alpha}_{a^+} f)(t) = t\frac{d}{dt}(H^{1-\alpha}_{a^+} f(t)) = \frac{1}{\Gamma(1-\alpha)} t\frac{d}{dt} \int_a^t \left(\log\frac{t}{s}\right)^{-\alpha} \frac{f(s)}{s} ds,$$

$$({}^H D^{\alpha}_{b^-} f)(t) = -t\frac{d}{dt}(H^{1-\alpha}_{b^-} f(t)) = \frac{-1}{\Gamma(1-\alpha)} t\frac{d}{dt} \int_t^b \left(\log\frac{s}{t}\right)^{-\alpha} \frac{f(s)}{s} ds,$$

with $t \in (a, b)$.

Katugampola introduced new fractional integral operators, called *Katugampola fractional integrals*, in the following way.

Definition 3. Let $0 < a < b$, $f : [a, b] \to \mathbb{R}$ an integrable function, and $\alpha \in (0, 1), \rho > 0$ two fixed real numbers. The *right and left side Katugampola fractional integrals of order* α are defined, respectively, by

$$K^{\alpha,\rho}_{a^+} f(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\alpha}} f(s) ds \tag{5}$$

and

$$K^{\alpha,\rho}_{b^-} f(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_t^b \frac{t^{\rho-1}}{(s^\rho - t^\rho)^{1-\alpha}} f(s) ds, \tag{6}$$

with $t \in (a, b)$.

Some generalizations of the Riemann-Liouville and Hadamard fractional derivatives appeared in [10]. These generalizations, called *Katugampola fractional derivatives*, are defined as

$$({}^K D^{\alpha,\rho}_{a^+} f)(t) = \frac{\rho^\alpha}{\Gamma(1-\alpha)}\ t^{1-\rho}\ \frac{d}{dt} \int_a^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^\alpha} f(s) ds,$$

$$({}^K D^{\alpha,\rho}_{b^-} f)(t) = \frac{-\rho^\alpha}{\Gamma(1-\alpha)} t^{1-\rho} \frac{d}{dt} \int_t^b \frac{s^{\rho-1}}{(s^\rho - t^\rho)^\alpha} f(s) ds,$$

with $t \in (a, b)$.

The relations between these two fractional operators are the following:

$$({}^K D^{\alpha,\rho}_{a^+} f)(t) = t^{1-\rho} \frac{d}{dt} K^{1-\alpha,\rho}_{a^+} f(t),$$

$$({}^K D^{\alpha,\rho}_{b^-} f)(t) = -t^{1-\rho} \frac{d}{dt} K^{1-\alpha,\rho}_{b^-} f(t).$$

Definition 4. Let $0 < a < b$, $g : [a, b] \to \mathbb{R}$ an increasing positive function on $(a, b]$ with continuous derivative on (a, b), $f : [a, b] \to \mathbb{R}$ an integrable function, and $\alpha \in (0, 1)$ a fixed real number. The right and left side fractional integrals in [11] of order α of f with respect to g are defined, respectively, by

$$I^\alpha_{g,a^+} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{g'(s) f(s)}{(g(t) - g(s))^{1-\alpha}} ds \tag{7}$$

and

$$I^\alpha_{g,b^-} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \frac{g'(s) f(s)}{(g(s) - g(t))^{1-\alpha}} ds, \tag{8}$$

with $t \in (a, b)$.

There are other definitions of integral operators in the global case, but they are slight modifications of the previous ones.

3 General fractional integral of Riemann-Liouville type

Now, we give the definition of a general fractional integral in [2].

Definition 5. Let $a < b$ and $\alpha \in \mathbb{R}^+$. Let $g : [a,b] \to \mathbb{R}$ be a positive function on $(a,b]$ with continuous positive derivative on (a,b) and $G : [0, g(b) - g(a)] \times (0,\infty) \to \mathbb{R}$ a continuous function which is positive on $(0, g(b) - g(a)] \times (0,\infty)$. Let us define the function $T : [a,b] \times [a,b] \times (0,\infty) \to \mathbb{R}$ by

$$T(t,s,\alpha) = \frac{G(|g(t) - g(s)|, \alpha)}{g'(s)}.$$

The *right and left integral operators*, denoted, respectively, by J^{α}_{T,a^+} and J^{α}_{T,b^-}, are defined for each measurable function f on $[a,b]$ as

$$J^{\alpha}_{T,a^+} f(t) = \int_a^t \frac{f(s)}{T(t,s,\alpha)} \mathrm{d}s, \tag{9}$$

$$J^{\alpha}_{T,b^-} f(t) = \int_t^b \frac{f(s)}{T(t,s,\alpha)} \mathrm{d}s, \tag{10}$$

with $t \in [a,b]$.

We say that $f \in L^1_T[a,b]$ if $J^{\alpha}_{T,a^+}|f|(t), J^{\alpha}_{T,b^-}|f|(t) < \infty$ for every $t \in [a,b]$.

Note that these operators generalize the integral operators in Definitions 1, 2, and 4:

(A) If we choose

$$g(t) = t,\ G(x,\alpha) = \Gamma(\alpha)\, x^{1-\alpha}, \quad T(t,s,\alpha) = \Gamma(\alpha)\, |t-s|^{1-\alpha},$$

then J^{α}_{T,a^+} and J^{α}_{T,b^-} are the right and left Riemann-Liouville fractional integrals ${}^{RL}J^{\alpha}_{a^+}$ and ${}^{RL}J^{\alpha}_{b^-}$ in (1) and (2), respectively. Its corresponding right and left Riemann-Liouville fractional derivatives are

$$({}^{RL}D^{\alpha}_{a^+} f)(t) = \frac{\mathrm{d}}{\mathrm{d}t}({}^{RL}J^{1-\alpha}_{a^+} f(t)) \quad \text{and} \quad ({}^{RL}D^{\alpha}_{b^-} f)(t) = -\frac{\mathrm{d}}{\mathrm{d}t}({}^{RL}J^{1-\alpha}_{b^-} f(t)).$$

(B) If we choose

$$g(t) = \log t, \quad G(x,\alpha) = \Gamma(\alpha) x^{1-\alpha}, \quad T(t,s,\alpha) = \Gamma(\alpha) t \left|\log\frac{t}{s}\right|^{1-\alpha},$$

then J^{α}_{T,a^+} and J^{α}_{T,b^-} are the right and left Hadamard fractional integrals $H^{\alpha}_{a^+}$ and $H^{\alpha}_{b^-}$ in (3) and (4), respectively. Its corresponding right and left Hadamard fractional derivatives are

$$({}^{H}D^{\alpha}_{a^+} f)(t) = t\frac{\mathrm{d}}{\mathrm{d}t}(H^{1-\alpha}_{a^+} f(t)), \quad \text{and} \quad ({}^{H}D^{\alpha}_{b^-} f)(t) = -t\frac{\mathrm{d}}{\mathrm{d}t}(H^{1-\alpha}_{b^-} f(t)).$$

(C) If we choose

$$g(t) = t^{\rho}, \quad G(x,\alpha) = \Gamma(\alpha)\rho^{\alpha} x^{1-\alpha}, \quad T(t,s,\alpha) = \frac{\Gamma(\alpha)}{\rho^{1-\alpha}} \frac{|t^{\rho} - s^{\rho}|^{1-\alpha}}{s^{\rho-1}},$$

then J^{α}_{T,a^+} and J^{α}_{T,b^-} are the right and left Katugampola fractional integrals $K^{\alpha,\rho}_{a^+}$ and $K^{\alpha,\rho}_{b^-}$ in (5) and (6), respectively. Its corresponding right and left Katugampola fractional derivatives are

$$({}^{K}D^{\alpha,\rho}_{a^+}f)(t) = t^{1-\rho}\frac{\mathrm{d}}{\mathrm{d}t}(K^{1-\alpha,\rho}_{a^+}f(t)), \quad \text{and} \quad ({}^{K}D^{\alpha,\rho}_{b^-}f)(t) = -t^{1-\rho}\frac{\mathrm{d}}{\mathrm{d}t}(K^{1-\alpha,\rho}_{b^-}f(t)).$$

(D) If we choose a function g with the properties in Definition 5 and

$$G(x,\alpha) = \Gamma(\alpha)x^{1-\alpha}, \quad \text{and} \quad T(t,s,\alpha) = \Gamma(\alpha)\frac{|g(t)-g(s)|^{1-\alpha}}{g'(s)},$$

then J^{α}_{T,a^+} and J^{α}_{T,b^-} are the right and left fractional integrals I^{α}_{g,a^+} and I^{α}_{g,b^-} in (7) and (8), respectively.

Definition 6. Let $a < b$ and $\alpha \in \mathbb{R}^+$. Let $g : [a,b] \to \mathbb{R}$ be a positive function on $(a,b]$ with continuous positive derivative on (a,b) and $G : [0, g(b) - g(a)] \times (0,\infty) \to \mathbb{R}$ a continuous function, which is positive on $(0, g(b) - g(a)] \times (0,\infty)$. For each function $f \in L^1_T[a,b]$, its *right and left generalized derivatives of order* α are defined, respectively, by

$$\begin{aligned} D^{\alpha}_{T,a^+}f(t) &= \frac{1}{g'(t)}\frac{\mathrm{d}}{\mathrm{d}t}(J^{1-\alpha}_{T,a^+}f(t)),\\ D^{\alpha}_{T,b^-}f(t) &= \frac{-1}{g'(t)}\frac{\mathrm{d}}{\mathrm{d}t}(J^{1-\alpha}_{T,b^-}f(t)). \end{aligned} \tag{11}$$

for each $t \in (a,b)$.

Note that if we choose

$$g(t) = t, \quad G(x,\alpha) = \Gamma(\alpha)x^{1-\alpha}, \quad T(t,s,\alpha) = \Gamma(\alpha)|t-s|^{1-\alpha},$$

then $D^{\alpha}_{T,a^+}f(t) = {}^{RL}D^{\alpha}_{a^+}f(t)$ and $D^{\alpha}_{T,b^-}f(t) = {}^{RL}D^{\alpha}_{b^-}f(t)$. Also, we can obtain Hadamard and other fractional derivatives as particular cases of this generalized derivative.

4 Jensen-type inequalities for *m*-convex functions

The property of m-convexity for functions on $[0,b]$, $b > 0$ was introduced in [30] as an intermediate property between the usual convexity and star-shaped property. Since then many properties, especially inequalities, have been obtained for them (cf. [5,12,15,22,31]). One of the classical integral inequalities frequently studied in this setting is Jensen's inequality, which relates the value of a convex function of an integral to the integral of the convex function. It was proved in 1906 [9], and it can be stated as follows:

Let μ be a probability measure on the space X. If $f : X \to (a,b)$ is μ-integrable and φ is a convex function on (a,b), then

$$\varphi\left(\int_X f\,\mathrm{d}\mu\right) \le \int_X \varphi \circ f\,\mathrm{d}\mu.$$

Definition 7. Let $I \subseteq \mathbb{R}$ be an interval containing zero, and let $m \in (0,1]$. A function $\varphi : I \to \mathbb{R}$ is said to be m-convex if the inequality

$$\varphi(tx + m(1-t)y) \le t\varphi(x) + m(1-t)\varphi(y) \tag{12}$$

holds for every pair of points $x, y \in I$ and every coefficient $t \in [0,1]$.

If $m \in (0,1)$, then the hypothesis $0 \in I$ guarantees that $tx + m(1-t)y \in I$.

It is clear that taking $m = 1$ in Definition 7 we recover the concept of classical convex functions on I. Note that in this case it is not necessary the hypothesis $0 \in I$, since $tx + (1-t)y \in I$ for every $x, y \in I$.

Note that if we choose the coefficient $t = 0$ in (12), we obtain the inequality $\varphi(my) \le m\varphi(y)$.

Also, Definition 7 is equivalent to

$$\varphi(mtx + (1-t)y) \le mt\varphi(x) + (1-t)\varphi(y), \tag{13}$$

for all $x, y \in I$ and $t \in [0, 1]$.

The following discrete Jensen-type inequality for m-convex functions was established in [22, Theorem 3.2]:

Theorem 1. *Let $I \subseteq \mathbb{R}$ be an interval containing the zero, and let $\sum_{k=1}^{n} w_k x_k$ be a convex combination of points $x_k \in I$ with coefficients $w_k \in [0, 1]$. If φ is an m-convex function on I, with $m \in (0, 1]$, then*

$$\varphi\left(m\sum_{k=1}^{n} w_k x_k\right) \le m\sum_{k=1}^{n} w_k \varphi(x_k). \tag{14}$$

This inequality is a discrete version of the following one for continuous m-convex functions [22, Corollary 4.2]:

Theorem 2. *Let μ be a probability measure on the space X. If $I \subseteq \mathbb{R}$ is an interval containing zero, $f: X \to I$ is μ-integrable, and φ is a continuous m-convex function on I, with $m \in (0, 1]$, then*

$$\varphi\left(m\int_X f \,\mathrm{d}\mu\right) \le m\int_X \varphi \circ f \,\mathrm{d}\mu. \tag{15}$$

The following discrete Jensen-type inequality for convex functions appears in [16, Theorem 1.2]:

Theorem 3. *Let $x_1 \le x_2 \le \cdots \le x_n$ and let $\{w_k\}_{k=1}^{n}$ be positive weights whose sum is 1. If φ is a convex function on $[x_1, x_n]$, then*

$$\varphi\left(x_1 + x_n - \sum_{k=1}^{n} w_k x_k\right) \le \varphi(x_1) + \varphi(x_n) - \sum_{k=1}^{n} w_k \varphi(x_k).$$

Our purpose is to prove continuous versions of the above discrete inequality in the setting of m-convexity (see Theorems 7 and 8). Before stating such a result, we require some properties of the m-convex functions.

Lemma 4. *Let $I \subseteq \mathbb{R}$ be an interval containing the zero, and let φ be an m-convex function on I with $m \in (0, 1]$. For $\{x_k\}_{k=1}^{n} \subset I$ such that $x_1 \le x_2 \le \cdots \le x_n$, the following inequalities hold:*

$$\varphi(x_1 + mx_n - mx_k) \le \varphi(x_1) + m\varphi(x_n) - \varphi(mx_k), \quad 1 \le k \le n. \tag{16}$$

Proof. Let us consider $y_k = x_1 + mx_n - mx_k$. Then $x_1 + mx_n = y_k + mx_k$ and so the pairs x_1, mx_n and y_k, mx_k have the same midpoint. Since $x_1 \le y_k$ and $mx_k \le mx_n$, we have $x_1 \le y_k, mx_k \le mx_n$ and there exists $\lambda \in [0, 1]$ such that

$$\begin{aligned} mx_k &= \lambda x_1 + m(1-\lambda)x_n, \\ y_k &= (1-\lambda)x_1 + m\lambda x_n, \end{aligned}$$

for $1 \le k \le n$. From Definition 7 and its equivalent form (13) we obtain

$$\begin{aligned} \varphi(y_k) &\le m\lambda\varphi(x_n) + (1-\lambda)\varphi(x_1) \\ &= \varphi(x_1) + m\varphi(x_n) - [\lambda\varphi(x_1) + m(1-\lambda)\varphi(x_n)] \\ &\le \varphi(x_1) + m\varphi(x_n) - \varphi(\lambda x_1 + m(1-\lambda)x_n) \\ &= \varphi(x_1) + m\varphi(x_n) - \varphi(mx_k), \end{aligned}$$

and (16) follows.

Note that since $m \in (0,1]$, the hypothesis $0 \in I$ guarantees that $mx_k \in I$. □

The following two results generalize Theorem 3 in the setting of m-convexity.

Theorem 5. *Let $I \subseteq \mathbb{R}$ be an interval containing the zero, let $\{x_k\}_{k=1}^n \subset I$ with $x_1 \le x_2 \le \cdots \le x_n$, and let $\{w_k\}_{k=1}^n$ be positive weights whose sum is* 1. *If φ is an m-convex function on I, with $m \in (0,1]$, then*

$$\varphi\left(mx_1 + m^2x_n - m^2\sum_{k=1}^n w_kx_k\right) \le m\varphi(x_1) + m^2\varphi(x_n) - m\sum_{k=1}^n w_k\varphi(mx_k). \tag{17}$$

Remark 1. Theorem 3 gives that if $m = 1$, the inequality in Theorem 5 also holds if we remove the hypothesis $0 \in I$.

Proof. First, note that

$$x_1 + mx_n - m\sum_{k=1}^n w_kx_k = \sum_{k=1}^n w_k(x_1 + mx_n - mx_k),$$

and thus

$$mx_1 + m^2x_n - m^2\sum_{k=1}^n w_kx_k = m\sum_{k=1}^n w_k(x_1 + mx_n - mx_k).$$

Then it follows from (14) and (16) that

$$\begin{aligned}\varphi\left(mx_1 + m^2x_n - m^2\sum_{k=1}^n w_kx_k\right) &= \varphi\left(m\sum_{k=1}^n w_k(x_1 + mx_n - mx_k)\right)\\ &\le m\sum_{k=1}^n w_k\varphi(x_1 + mx_n - mx_k)\\ &\le m\sum_{k=1}^n w_k(\varphi(x_1) + m\varphi(x_n) - \varphi(mx_k))\\ &= m\varphi(x_1) + m^2\varphi(x_n) - m\sum_{k=1}^n w_k\varphi(mx_k),\end{aligned}$$

and this concludes the proof of the inequality.

Let us check that the hypothesis $0 \in I$ guarantees that $mx_1 + m^2x_n - m^2\sum_{k=1}^n w_kx_k \in I$:

Assume that $I = [a,b]$. Then

$$mx_1 + m^2x_n - m^2\sum_{k=1}^n w_kx_k \ge mx_1 + m^2x_n - m^2\sum_{k=1}^n w_kx_n = mx_1 \ge \min\{0, x_1\} \ge a.$$

Also,

$$mx_1 + m^2x_n - m^2\sum_{k=1}^n w_kx_k \le mx_1 + m^2x_n - m^2\sum_{k=1}^n w_kx_1 = mx_1 - m^2x_1 + m^2x_n.$$

If $x_1 \le 0$, then $mx_1 - m^2x_1 \le 0$ and so

$$mx_1 + m^2x_n - m^2\sum_{k=1}^n w_kx_k \le mx_1 - m^2x_1 + m^2x_n \le m^2x_n \le \max\{0, x_n\} \le b.$$

Assume now that $x_1 > 0$. Let us consider the function $v(t) = tx_1 - t^2x_1 + t^2x_n$. Since $v'(t) = x_1 + 2t(x_n - x_1) > 0$ and $m \in (0,1]$, we have

$$v(m) \le v(1) = x_n \le b, \quad mx_1 + m^2x_n - m^2\sum_{k=1}^n w_kx_k \le mx_1 - m^2x_1 + m^2x_n = v(m) \le b.$$

If I is not a closed interval $[a, b]$, a similar argument gives the result. □

If φ is a continuous m-convex function, we can obtain the following improvement of Theorem 5.

Theorem 6. *Let $a \le 0 \le b$, let $\{y_k\}_{k=1}^n \subset [a, b]$, and let $\{w_k\}_{k=1}^n$ be positive weights whose sum is 1. If φ is a continuous m-convex function on $[a, b]$, with $m \in (0, 1]$, then*

$$\varphi\left(ma + m^2 b - m^2 \sum_{k=1}^n w_k y_k\right) \le m\varphi(a) + m^2\varphi(b) - m\sum_{k=1}^n w_k \varphi(my_k). \tag{18}$$

Proof. If we consider $0 < \varepsilon < 1$, $y_0 = a$, $y_{n+1} = b$, $w_k' = (1-\varepsilon)w_k (1 \le k \le n)$, $w_0' = \varepsilon/2$, and $w_{n+1}' = \varepsilon/2$, then $\sum_{k=0}^{n+1} w_k' = 1$ and Theorem 5 gives

$$\begin{aligned}\varphi\left(ma + m^2 b - \frac{m^2\varepsilon}{2}a - \frac{m^2\varepsilon}{2}b - m^2\sum_{k=1}^n (1-\varepsilon)w_k y_k\right)\\ \le m\varphi(a) + m^2\varphi(b) - \frac{m\varepsilon}{2}\varphi(ma) - \frac{m\varepsilon}{2}\varphi(mb) - m\sum_{k=1}^n (1-\varepsilon)w_k\varphi(my_k).\end{aligned}$$

Since φ is a continuous function on $[a, b]$, if we take $\varepsilon \to 0^+$, we obtain (18). □

Next, we present a continuous version of the above discrete inequality.

Theorem 7. *Let μ be a probability measure on the space X and $a \le 0 \le b$ real constants. If $f : X \to [a, b]$ is a measurable function and φ is a continuous m-convex function on $[a, b]$, with $m \in (0, 1]$, then f and $\varphi(mf)$ are μ-integrable functions and*

$$\varphi\left(ma + m^2 b - m^2\int_X f\,\mathrm{d}\mu\right) \le m\varphi(a) + m^2\varphi(b) - m\int_X \varphi(mf)\,\mathrm{d}\mu. \tag{19}$$

If $m = 1$, this inequality also holds if we remove the hypothesis $0 \in [a, b]$.

Proof. Since $a \le f \le b$ and φ is a continuous function on $[a, b]$, we have that f and $\varphi(mf)$ are bounded measurable functions on X. And using that μ is a probability measure on X, we conclude that f and $\varphi(mf)$ are μ-integrable functions.

For each $n \ge 1$ and $0 \le k \le 2^n$, let us consider the sets

$$E_{n,k} = \{x \in X : a + k2^{-n}(b-a) \le f(x) < a + (k+1)2^{-n}(b-a)\}.$$

Since f is a measurable function satisfying $a \le f \le b$, we have that $\{E_{n,k}\}_{k=0}^{2^n}$ are pairwise disjoint measurable sets and $X = \cup_{k=0}^{2^n} E_{n,k}$ for each n. Thus,

$$\sum_{k=0}^{2^n} \mu(E_{n,k}) = 1$$

for each n.

Since f is a measurable function satisfying $a \le f \le b$ and $\{E_{n,k}\}_{k=0}^{2^n}$ is a partition of X, the sequence of simple functions

$$f_n = \sum_{k=0}^{2^n} (a + k2^{-n}(b-a))\chi_{E_{n,k}}$$

satisfies $a \le f_n \le b$ and $f - 2^{-n}(b-a) < f_n \le f$ for every n and so

$$\lim_{n\to\infty} f_n = f.$$

Note that

$$\int_X f_n \mathrm{d}\mu = \sum_{k=0}^{2^n} (a + k2^{-n}(b-a))\mu(E_{n,k}).$$

Since $\{E_{n,k}\}_{k=0}^{2^n}$ is a partition of X, we have

$$\varphi(mf) = \sum_{k=0}^{2^n} \varphi(ma + mk2^{-n}(b-a))\chi_{E_{n,k}}, \quad \int_X \varphi(mf)\mathrm{d}\mu = \sum_{k=0}^{2^n} \varphi(ma + mk2^{-n}(b-a))\mu(E_{n,k}).$$

Hence, Theorem 6 gives

$$\varphi\left(ma + m^2 b - m^2 \int_X f_n \mathrm{d}\mu\right) \le m\varphi(a) + m^2\varphi(b) - m\int_X \varphi(mf)\mathrm{d}\mu. \tag{20}$$

If $m = 1$, Theorem 3 gives the above inequality without the hypothesis $0 \in [a, b]$.

Since $a \le f_n \le b$ for every n, μ is a finite measure and $\lim_{n\to\infty} f_n = f$, dominated convergence theorem gives

$$\lim_{n\to\infty} \int_X f_n \mathrm{d}\mu = \int_X f \mathrm{d}\mu.$$

If $m \in (0, 1)$, then the hypothesis $0 \in [a, b]$ guarantees that $ma + m^2 b - m^2 \int_X f_n \mathrm{d}\mu \in [a, b]$:

Since $a \le 0 \le b$, we have

$$ma + m^2 b - m^2 \int_X f_n \mathrm{d}\mu \le ma + m^2 b - m^2 a = m(1-m)a + m^2 b \le m^2 b \le b,$$

$$ma + m^2 b - m^2 \int_X f_n \mathrm{d}\mu \ge ma + m^2 b - m^2 b = ma \ge a.$$

If $m = 1$, then

$$a + b - \int_X f_n \mathrm{d}\mu \le a + b - a = b, \quad a + b - \int_X f_n \mathrm{d}\mu \ge a + b - b = a,$$

and so, we do not need the hypothesis $0 \in [a, b]$.

Since

$$a \le ma + m^2 b - m^2 \int_X f_n \mathrm{d}\mu \le b$$

for every n and $m \in (0, 1]$, and φ is a continuous function on $[a, b]$,

$$\lim_{n\to\infty} \varphi\left(ma + m^2 b - m^2 \int_X f_n \mathrm{d}\mu\right) = \varphi\left(ma + m^2 b - m^2 \int_X f \mathrm{d}\mu\right).$$

Since $a \le mf_n \le b$ for every n, $\lim_{n\to\infty} f_n = f$ and φ is a continuous function on $[a, b]$, $\lim_{n\to\infty} \varphi(mf_n) = \varphi(mf)$.

Again, from the continuity of φ on $[a, b]$, there exists a positive constant K with $|\varphi| \le K$ on $[a, b]$ and so, $|\varphi(mf_n)| \le K$ for every n.

In view of the finiteness of μ, dominated convergence theorem guarantees that

$$\lim_{n\to\infty}\int_X \varphi(mf)\mathrm{d}\mu = \int_X \varphi(mf)\mathrm{d}\mu.$$

Combining the foregoing facts with (20), we obtain (19). □

If $m = 1$, it is possible to improve Theorem 7, by removing the hypothesis of continuity.

Theorem 8. *Let μ be a probability measure on the space X and $a \le b$ real constants. If $f : X \to [a, b]$ is a measurable function and φ is a convex function on $[a, b]$, then f and $\varphi \circ f$ are μ-integrable functions and*

$$\varphi\left(a + b - \int_X f\mathrm{d}\mu\right) \le \varphi(a) + \varphi(b) - \int_X \varphi \circ f\mathrm{d}\mu.$$

Proof. Since φ is a convex function on $[a, b]$, φ is continuous on (a, b) and there exist the limits

$$\lim_{s\to a^+}\varphi(s), \quad \text{and} \quad \lim_{s\to b^-}\varphi(s).$$

Define φ^* as follows:

$$\varphi^*(t) = \begin{cases} \varphi(t) & \text{if } t \in (a, b), \\ \lim\limits_{s\to a^+}\varphi(s) & \text{if } t = a, \\ \lim\limits_{s\to b^-}\varphi(s) & \text{if } t = b. \end{cases}$$

Hence, φ^* is a continuous convex function on $[a, b]$, and by Theorem 7 with $m = 1$, we have

$$\varphi^*\left(a + b - \int_X f\mathrm{d}\mu\right) \le \varphi^*(a) + \varphi^*(b) - \int_X \varphi^* \circ f\mathrm{d}\mu.$$

Assume that $f = a$ μ-a.e. or $f = b$ μ-a.e.; in the first case,

$$\varphi\left(a + b - \int_X f\mathrm{d}\mu\right) = \varphi(a + b - a) = \varphi(b) = \varphi(a) + \varphi(b) - \varphi(a) = \varphi(a) + \varphi(b) - \int_X \varphi \circ f\mathrm{d}\mu;$$

in the second case,

$$\varphi\left(a + b - \int_X f\mathrm{d}\mu\right) = \varphi(a + b - b) = \varphi(a) = \varphi(a) + \varphi(b) - \varphi(b) = \varphi(a) + \varphi(b) - \int_X \varphi \circ f\mathrm{d}\mu.$$

Otherwise, $a < \int_X f\mathrm{d}\mu < b$ and $a < a + b - \int_X f\mathrm{d}\mu < b$. Consequently,

$$\varphi\left(a + b - \int_X f\mathrm{d}\mu\right) = \varphi^*\left(a + b - \int_X f\mathrm{d}\mu\right).$$

If we define

$$\Delta_a = \varphi(a) - \varphi^*(a) \ge 0, \quad \Delta_b = \varphi(b) - \varphi^*(b) \ge 0,$$

then

$$\varphi = \varphi^* + \Delta_a\chi_{\{a\}} + \Delta_b\chi_{\{b\}}, \quad \varphi \circ f = \varphi^* \circ f + \Delta_a \quad \chi_{\{f=a\}} + \Delta_b \quad \chi_{\{f=b\}},$$

where χ_A is the function with value 1 on the set A and 0 otherwise (i.e., the characteristic function of A).

Hence,

$$\int_X \varphi \circ f \mathrm{d}\mu = \int_X \varphi^* \circ f \mathrm{d}\mu + \Delta_a \mu(\{f = a\}) + \Delta_b \mu(\{f = b\}),$$

and we have

$$\begin{aligned}
\varphi\left(a + b - \int_X f \mathrm{d}\mu\right) = \varphi^*\left(a + b - \int_X f \mathrm{d}\mu\right) &\le \varphi^*(a) + \varphi^*(b) - \int_X \varphi^* \circ f \mathrm{d}\mu \\
&= \varphi(a) - \Delta_a + \varphi(b) - \Delta_b - \int_X \varphi \circ f \mathrm{d}\mu + \Delta_a \mu(\{f = a\}) + \Delta_b \mu(\{f = b\}) \\
&= \varphi(a) + \varphi(b) - \int_X \varphi \circ f \mathrm{d}\mu - \Delta_a[1 - \mu(\{f = a\})] - \Delta_b[1 - \mu(\{f = b\})] \\
&\le \varphi(a) + \varphi(b) - \int_X \varphi \circ f \mathrm{d}\mu.
\end{aligned}$$

□

Theorem 8 has the following direct consequence.

Corollary 9. *Let $a \le b$, let $\{y_k\}_{k=1}^n \subset [a, b]$, and let $\{w_k\}_{k=1}^n$ be positive weights whose sum is 1. If φ is a convex function on $[a, b]$, then*

$$\varphi\left(a + b - \sum_{k=1}^n w_k y_k\right) \le \varphi(a) + \varphi(b) - \sum_{k=1}^n w_k \varphi(y_k).$$

Note that Theorem 8 provides a kind of converse of the classical Jensen's inequality for convex functions.

Proposition 10. *Let μ be a probability measure on the space X and $a \le b$ real constants. If $f : X \to [a, b]$ is a measurable function and φ is a convex function on $[a, b]$, then f and $\varphi \circ f$ are μ-integrable functions and*

$$\varphi\left(\int_X f \mathrm{d}\mu\right) \le \int_X \varphi \circ f \mathrm{d}\mu \le \varphi(a) + \varphi(b) - \varphi\left(a + b - \int_X f \mathrm{d}\mu\right).$$

Theorems 8 and 7 have, respectively, the following direct consequences for general fractional integrals of Riemann-Liouville type.

Proposition 11. *Let $c < d$ and $a \le b$ be real constants. If $f : [c, d] \to [a, b]$ is a measurable function, φ is a convex function on $[a, b]$, and*

$$\mathbb{T}(\alpha) = \int_c^d \frac{1}{T(d, s, \alpha)} \mathrm{d}s = \int_0^{g(d)-g(c)} \frac{\mathrm{d}x}{G(x, \alpha)} < \infty,$$

then $f(s)/T(d, s, \alpha)$, $\varphi(f(s))/T(d, s, \alpha) \in L^1[c, d]$, and

$$\varphi\left(a + b - \frac{1}{\mathbb{T}(\alpha)} \int_c^d \frac{f(s)}{T(d, s, \alpha)} \mathrm{d}s\right) \le \varphi(a) + \varphi(b) - \frac{1}{\mathbb{T}(\alpha)} \int_c^d \frac{\varphi(f(s))}{T(d, s, \alpha)} \mathrm{d}s.$$

Proposition 12. *Let $c < d$ and $a \le 0 \le b$ be real constants. If $f : [c, d] \to [a, b]$ is a measurable function, φ is a continuous m-convex function on $[a, b]$, with $m \in (0, 1]$, and*

$$\mathbb{T}(\alpha) = \int_c^d \frac{1}{T(d, s, \alpha)} \mathrm{d}s = \int_0^{g(d)-g(c)} \frac{\mathrm{d}x}{G(x, \alpha)} < \infty,$$

then $f(s)/T(d, s, \alpha)$, $\varphi(mf(s))/T(d, s, \alpha) \in L^1[c, d]$ and

$$\varphi\left(ma + m^2 b - \frac{m^2}{\mathbb{T}(\alpha)} \int_c^d \frac{f(s)}{T(d, s, \alpha)} \mathrm{d}s\right) \le m\varphi(a) + m^2\varphi(b) - \frac{m}{\mathbb{T}(\alpha)} \int_c^d \frac{\varphi(mf(s))}{T(d, s, \alpha)} \mathrm{d}s.$$

5 Conclusion

The main goal of our research is to determine new Jensen-type inequalities for m-convex functions and apply them to generalized Riemann-Liouville-type integral operators. In particular, we prove continuous versions of the discrete Jensen-type inequality in [16, Theorem 1.2], in the setting of m-convexity. Furthermore, our results provide new inequalities for convex functions.

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